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Published in:
International Journal of Modern Physics A

DOI:
10.1142/S0217751X95001662

International Journal of Modern Physics A, Vol. 10, No. 24 (1995) 3477–3496

THE CALOGERO MODEL AND THE VIRASORO SYMMETRY

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Received 22 December 1994

We construct new realizations of the Virasoro algebra inspired by the Calogero model. The Virasoro algebra we find acts as a kind of spectrum-generating algebra of the Calogero model. Furthermore, we present the superextension of these results and introduce a class of higher spin extensions of the Virasoro algebra which are of the W_∞ type.

1. Introduction

Over the years there has been an increasing interest in the possible relationships between integrable systems and conformal field theory. A well-known and well-studied example is provided by the Liouville model and, more generally, the Toda models which play a crucial role in the study of noncritical string theories.

Another example of an integrable system is the Calogero model, which has been studied intensely since its construction in 1969.[1,2] Recently, further progress has been obtained in understanding the N-body Calogero model with harmonic interaction.[3,4] In Refs. 5–7 it was argued that the Calogero model describes one-dimensional reductions of anyonic systems,[8,9] for example anyons at the lowest Landau level in a strong magnetic field. In its turn, anyon physics plays an important role in the understanding of the (fractional) quantum Hall effect (QHE),[10] which assumes interesting links between the latter and the Calogero model.

The recent progress[3,4] in the understanding of the (rational) Calogero model was based on an extension of the Heisenberg algebra, the so-called $\mathrm{SH}_N(\nu)$ algebra.[11] For $N = 1$ the algebra reduces to the ordinary Heisenberg algebra which underlies the higher spin algebras in higher dimensions[12] as well as the standard realization of the Virasoro algebra as vector fields and of the $W_{1+\infty}$ algebra[13] as differential operators on the circle.

The question we address in this paper is whether the general $\mathrm{SH}_N(\nu)$ algebra can lead to new realizations of the (super) Virasoro algebra. We show that this is

*Supported in part by the Russian Fund of Fundamental Research, grant N67123016.

indeed the case and, as a result, one discovers a new interesting class of W_∞ type algebras containing higher spin currents in addition to the Virasoro generators. We expect that our results will lead to a better insight into possible relations between the Calogero model and conformal field theory. Our results may also have applications to the QHE. Actually, it was observed recently[14] that the relevant approach to the QHE is based on the representation theory of the $W_{1+\infty}$ algebra developed in Ref. 15. Since, on the other hand, the algebras considered in the present paper are responsible for the N-body excitations in the Calogero model and can be regarded as some (extensions $\times$ deformations) of the ordinary $W_{1+\infty}$ algebra, one can speculate that they might be relevant to the analysis of the many-body excitations in the QHE.

2. The $\mathrm{SH}_N(\nu)$ Realization of the Virasoro Algebra

Our starting point is the S_N-extended Heisenberg algebra $\mathrm{SH}_N(\nu)$,[3,4,11] which can be regarded as the algebra formed by the generating elements $a_i, a_i^\dagger$ and K_{ij} $(i,j = 1,\ldots,N)$ obeying the relations[a]

$$[a_i^{(\dagger)}, a_j^{(\dagger)}] = 0\,, \quad [a_i, a_j^\dagger] = A_{ij} \equiv \delta_{ij}\left(1 + \nu\sum_{l=1}^{N} K_{il}\right) - \nu K_{ij}\,, \tag{1}$$

$$K_{ij}K_{jl} = K_{jl}K_{il} = K_{il}K_{ij}\,, \quad \text{for all } i \neq j,\ i \neq l,\ j \neq l\,, \tag{2}$$

$$(K_{ij})^2 = I\,, \qquad K_{ij} = K_{ji}\,, \tag{3}$$

$$K_{ij}K_{mn} = K_{mn}K_{ij}\,, \quad \text{if all indices } i,j,m,n \text{ different}\,, \tag{4}$$

$$K_{ij}a_j^{(\dagger)} = a_i^{(\dagger)}K_{ij}\,. \tag{5}$$

Here ν is a constant related to the Calogero coupling constant, while K_{ij} are the elementary permutation operators of the S_N exchange algebra. We use the standard convention that square brackets denote commutators and curly brackets anticommutators.

To make contact between the $\mathrm{SH}_N(\nu)$ algebra and the Calogero model,[b] one has to use the following "Calogero realization" of $\mathrm{SH}_N(\nu)$:

$$a_i = \frac{1}{\sqrt{2}}(x_i + D_i)\,, \quad a_i^\dagger = \frac{1}{\sqrt{2}}(x_i - D_i)\,, \tag{6}$$

with

$$D_i = \frac{\partial}{\partial x_i} + \nu\sum_{j\neq i}(x_i - x_j)^{-1}(1 - K_{ij})\,. \tag{7}$$

[a]In this paper repeated indices do not imply summation.

[b]For a review of the Calogero model as a classical and quantum integrable model, see Ref. 16.

The real Calogero coordinates x_i and the so-called Dunkl derivatives D_i [17] can be shown, by a direct calculation, to satisfy the commutation relations[3,4]

$$[x_i, x_j] = [D_i, D_j] = 0\,, \quad [D_i, x_j] = A_{ij}\,. \tag{8}$$

The crucial observation that establishes the relation with the Calogero model is that the Calogero Hamiltonian $H_{\rm Cal}$ is related to the operator

$$H = \frac{1}{2}\sum_{i=1}^{N}\{a_i\,, a_i^\dagger\}\,. \tag{9}$$

This operator, like the ordinary harmonic oscillator Hamiltonian, obeys the standard relations

$$[H, a_i^\dagger] = a_i^\dagger\,, \quad [H, a_i] = -a_i\,. \tag{10}$$

The operator H is called the universal Calogero Hamiltonian, because it also can be used to describe spinning Calogero models.[18,19] The precise relationship with the original Calogero Hamiltonian $H_{\rm Cal}$ involves a simple similarity transformation followed by a restriction to the subspace of totally symmetric wave functions. This construction allows one to construct all eigen-wave-functions of the model as the Fock vectors

$$(a_1^\dagger)^{n_1}\cdots(a_N^\dagger)^{n_N}|0\rangle\,. \tag{11}$$

For more details, see Ref. 4.

Our goal in this section is to investigate whether the $\mathrm{SH}_N(\nu)$ algebra defined in Eqs. (1)–(5) can lead to new realizations of the Virasoro algebra. Our starting point is the following ansatz for the Virasoro generators:

$$L_{-n} = \sum_{i=1}^{N}\left[\alpha(a_i^\dagger)^{n+1}a_i + \beta a_i(a_i^\dagger)^{n+1} + \left(\lambda - \frac{1}{2}\right)(n+1)(a_i^\dagger)^n\right], \tag{12}$$

where α, β and λ are arbitrary parameters.[c] Note that for a vanishing value of the parameter ν we have $[a_i, a_j^\dagger] = \delta_{ij}$ and the above generators correspond to a direct sum of the N standard vector field generators of the Virasoro algebra. Remarkably, it turns out that the ansatz (12) also works for nonvanishing values of ν. In order to prove this it is convenient to sum over all the modes and to rewrite the ansatz for the Virasoro generators in terms of a parameter function $\xi(a_i^\dagger)$ as

$$L_\xi = \sum_{i=1}^{N}\left[\alpha\ \xi(a_i^\dagger)a_i + \beta\ a_i\xi(a_i^\dagger) + \left(\lambda - \frac{1}{2}\right)\frac{\partial}{\partial a_i^\dagger}\xi(a_i^\dagger)\right]. \tag{13}$$

[c] For $N = 1$ and $\alpha = \beta = 1/2$ the parameter λ coincides with the λ of Ref. 20.

The proof that the ansatz (13) satisfies the Virasoro algebra commutation relations requires the three identities

$$\sum_{i=1}^{N} \xi_1(a_i^\dagger)[a_i, \xi_2(a_j^\dagger)] - (1 \leftrightarrow 2) = \xi_1(a_j^\dagger)\frac{\partial}{\partial a_j^\dagger}\xi_2(a_j^\dagger) - (1 \leftrightarrow 2)\,, \tag{14}$$

$$\sum_{i=1}^{N} [a_i, \xi_1(a_j^\dagger)]\xi_2(a_i^\dagger) - (1 \leftrightarrow 2) = \frac{\partial}{\partial a_j^\dagger}\xi_1(a_j^\dagger)\,\xi_2(a_j^\dagger) - (1 \leftrightarrow 2)\,, \tag{15}$$

$$\sum_{i,j=1}^{N} \Big[[a_i, \xi_1(a_i^\dagger)], [a_j, \xi_2(a_j^\dagger)]\Big] = 0\,, \tag{16}$$

where ξ_1 and ξ_2 are arbitrary Laurent series. The proof of the identities (14)–(16) is given in App. A. It is based on the useful formula

$$[a_i, f(a^\dagger)] = \frac{\partial}{\partial a_i^\dagger} f(a^\dagger) - \nu \sum_{l=1}^{N} (a_i^\dagger - a_l^\dagger)^{-1} [K_{il}, f(a^\dagger)]\,. \tag{17}$$

This formula is a direct consequence of the basic commutation relations (1) and can be easily proven by expanding $f(a^\dagger)$ in a Laurent series in $a_i^\dagger$. Alternatively, one can observe that the right hand side of (17) (i) respects the Leibniz rule, (ii) vanishes when $f =$ const and (iii) reduces to the basic commutation relation (1) for $f = a_i^\dagger$. Combined together the properties (i)–(iii) prove that the formula (17) is valid for an arbitrary polynomial in $a_i^\dagger$ and $(a_i^\dagger)^{-1}$.

Note that the right hand side of (17) remains regular for regular functions $f(a^\dagger)$, i.e. the poles in $(a_i^\dagger - a_l^\dagger)$ cancel due to the commutator with K_{il}. It is worth mentioning that the formula (17) can be used for a simple derivation of the Dunkl derivative and its further generalizations given in Ref. 11 as the action of the a_i type operators in the Fock modules with the vacuum states satisfying the conditions $a_i|0\rangle = 0$ and $K_{ij}|0\rangle = T_{ij}|0\rangle$, where the matrices T_{ij} realize some representation t of the symmetric group S_N which acts on the vacuum vector(s) $|0\rangle$. The Dunkl derivatives then correspond to the trivial representation of S_N, $T_{ij} = 1$, while the derivatives introduced in Ref. 11 correspond to a general representation t.

We now proceed with the proof that the ansatz (13) satisfies the Virasoro algebra commutation relations. For simplicity, we first consider the special case with $\lambda = 1/2$. Using the standard Leibniz rule we find that

$$\begin{aligned}[L_{\xi_1}, L_{\xi_2}] = \sum_{i,j=1}^{N} \Big\{&\alpha^2 \xi_1(a_i^\dagger)[a_i, \xi_2(a_j^\dagger)]a_j - \beta^2 a_j [a_i, \xi_1(a_j^\dagger)]\xi_2(a_i^\dagger) \\ &+ \alpha\beta[\xi_1(a_i^\dagger)a_i, a_j \xi_2(a_j^\dagger)]\Big\} - (1 \leftrightarrow 2)\,. \end{aligned} \tag{18}$$

The third term on the right hand side can be rewritten as

$$\alpha\beta \sum_{i,j=1}^{N} \left\{ \frac{1}{2}[\xi_1(a_i^\dagger)a_i, \xi_2(a_j^\dagger)a_j] + \frac{1}{2}[a_j\xi_1(a_j^\dagger), a_i\xi_2(a_i^\dagger)] \right.$$
$$\left. - \Big[[a_i, \xi_1(a_i^\dagger)], [a_j, \xi_2(a_j^\dagger)]\Big] \right\} - (1 \leftrightarrow 2)\,. \qquad (19)$$

The last term in this expression vanishes due to the identity (16) while the first two terms are identical to the contribution from the α^2 and β^2 terms. Application of the identities (14) and (15) to the remaining terms gives the result

$$[L_{\xi_1}, L_{\xi_2}] = (\alpha+\beta)\sum_{i=1}^{N}[\alpha\xi_{1,2}(a_i^\dagger)a_i + \beta a_i \xi_{1,2}(a_i^\dagger)]\,, \qquad (20)$$

with the parameter $\xi_{1,2}(a_i^\dagger)$ given by

$$\xi_{1,2}(a_i^\dagger) = \xi_1(a_i^\dagger)\frac{\partial}{\partial a_i^\dagger}\xi_2(a_i^\dagger) - \xi_2(a_i^\dagger)\frac{\partial}{\partial a_i^\dagger}\xi_1(a_i^\dagger)\,. \qquad (21)$$

We conclude that the ansatz (12) leads to the Virasoro algebra provided that

$$\alpha + \beta = 1\,. \qquad (22)$$

This result can easily be extended to other values of λ with $\lambda \neq 1/2$ by the use of only the identities (14) and (15).

Thus it is shown that there exists a two-parameter class[d] of realizations of the Virasoro algebra constructed from the Calogero oscillators which are the generating elements of $\mathrm{SH}_N(\nu)$. We now make some comments on the above result:

(1) The proof that the generators (13) form the Virasoro algebra is based entirely on the commutation relations (1)–(5) and is independent of any particular realization of this algebra. By their construction the generators are invariant under the action of the symmetric group

$$K_{ij}L_\xi = L_\xi K_{ij}\,. \qquad (23)$$

Among other things this means that the Virasoro algebra closes and the above S_N invariance property remains valid if the parameters α and λ are not just pure numbers but depend on any combination of group algebra elements of S_N which are S_N-invariant themselves, i.e.

$$\alpha = \alpha(T_n)\,, \quad \lambda = \lambda(T_n)\,, \qquad (24)$$

[d] Particular examples of this class have been known to other authors: S. Isakov and J. Leinaas have studied the case with $\alpha = \beta = 1/2$, while the fact that the Virasoro algebra closes for the cases with $\alpha = 0$ or $\beta = 0$ was pointed out to us by A. Polychronakos (private communications).

$$T_n = \sum_{i_1 \neq i_2 \neq \cdots \neq i_n} K_{i_1 i_2} K_{i_2 i_3} \cdots K_{i_n i_1} \,. \tag{25}$$

In the discussion below we will need only the case where α is constant and λ is at most linear in K_{ij}:

$$\alpha = \alpha_0 \,, \quad \lambda = \lambda_0 + \lambda_1 \sum_{i \neq j} K_{ij} \,, \tag{26}$$

where α_0, λ_0 and λ_1 are some constants. This freedom allows us to define the L_0 generator of the Virasoro algebra in such a way that it satisfies the properties of a Z grading operator for the whole enveloping algebra of $\mathrm{SH}_N(\nu)$, i.e. it satisfies the same identities as the universal Calogero Hamiltonian [see (10)].

The generators $\{L_1, L_0, L_{-1}\}$, which are given by

$$\begin{aligned} L_1 &= \sum_{i=1}^{N} a_i \,, \\ L_0 &= \sum_{i=1}^{N} \left[\alpha a_i^\dagger a_i + (1-\alpha) a_i a_i^\dagger\right] + \left(\lambda - \frac{1}{2}\right) N \,, \\ L_{-1} &= \sum_{i=1}^{N} \left[\alpha (a_i^\dagger)^2 a_i + (1-\alpha) a_i (a_i^\dagger)^2\right] + 2\left(\lambda - \frac{1}{2}\right) \sum_{i=1}^{N} a_i^\dagger \,, \end{aligned} \tag{27}$$

form an sl(2) subalgebra of the Virasoro algebra. L_1 coincides with the annihilation operator of the center of mass degree of freedom while L_0 is nothing else than the universal Calogero Hamiltonian (9) modulo terms which become a constant when acting on the subspace of symmetric wave functions, i.e.

$$L_0 - H = (\lambda - \alpha) N - \nu \left(\alpha - \frac{1}{2}\right) \sum_{i \neq j} K_{ij} \,. \tag{28}$$

One can fix the free parameters α and λ to be

$$\alpha = \alpha_0 \qquad \lambda = \lambda_0 + \nu N^{-1} \left(\alpha - \frac{1}{2}\right) \sum_{i \neq j} K_{ij} \,, \tag{29}$$

so that $L_0 - H$ becomes a pure constant:

$$L_0 - H = (\lambda_0 - \alpha_0) N \,. \tag{30}$$

We denote the sl(2) subalgebra spanned by $\{L_{-1}, L_0, L_1\}$ as the "horizontal" sl(2) algebra to distinguish it from the "vertical" sl(2) algebra of Ref. 21, which also acts on the states of the Calogero model and is spanned by the generators

$$B_2^+ = \frac{1}{2} \sum_{i=1}^{N} (a_i^\dagger)^2 \,, \quad B_2^0 = H \,, \quad B_2^- = \frac{1}{2} \sum_{i=1}^{N} (a_i)^2 \,. \tag{31}$$

Remarkably, the (shifted by a constant) Calogero Hamiltonian serves as the Cartan subalgebra generator of both sl(2) algebras. We have not been able to extend the vertical sl(2) algebra to a Virasoro algebra.

The excited states of the Calogero model can be classified with respect to both "vertical" and "horizontal" sl(2) algebras. Our results imply that for the latter case these representations extend to appropriate representations of the whole reductive part of the Virasoro algebra spanned by the L_n with $n \leq 1$. It is worth mentioning that both the "vertical" sl(2) algebra and the "horizontal" Virasoro algebra are generated by symmetric combinations of the Calogero creation and annihilation operators a_i and $a_i^\dagger$ [cf. (23)] so that the Virasoro algebra under consideration does indeed leave invariant the subspace of totally symmetric wave functions of the Calogero model.

(2) So far we have used the Calogero realization where the creation and annihilation operators are expressed in terms of the real coordinates x_i underlying the Calogero model. However, one can use other representations of the same algebra equally well. For example, the Virasoro commutation relations remain valid for all other representations found in Ref. 11. A particularly useful realization is that which for $\nu = 0$ reduces to the standard holomorphic representation and thus can be expected to be relevant to conformal field theory. To be specific, consider the "holomorphic realization" of the $\mathrm{SH}_N(\nu)$ algebra given by

$$a_i = D^z{}_i\,, \quad a_i^\dagger = z_i\,, \tag{32}$$

where z_i are N complex coordinates and $D^z{}_i$ is the complex Dunkl derivative:

$$D^z{}_i = \frac{\partial}{\partial z_i} + \nu \sum_{j \neq i} (z_i - z_j)^{-1}(1 - K_{ij})\,. \tag{33}$$

In the holomorphic representation the Virasoro generators take the form

$$L_\xi = \sum_{i=1}^{N} \left[\alpha \xi(z_i) D^z{}_i + (1-\alpha) D^z{}_i \xi(z_i) + \left(\lambda - \frac{1}{2}\right) \frac{\partial}{\partial z_i} \xi(z_i) \right]. \tag{34}$$

Inserting (33) into (34), one can write L_ξ as follows:

$$\begin{aligned} L_\xi = {} & \sum_{i=1}^{N} \left[\alpha \xi(z_i) \partial_i^{\mathrm{KZ}} + (1-\alpha) \partial_i^{\mathrm{KZ}} \xi(z_i) + \left(\lambda - \frac{1}{2}\right) \frac{\partial}{\partial z_i} \xi(z_i) \right] \\ & + \nu(1-2\alpha) \sum_{i \neq j} \xi(z_i) \frac{1}{z_i - z_j} K_{ij}\,, \end{aligned} \tag{35}$$

where the Knizhnik–Zamolodchikov type derivatives ∂_i^{KZ} are defined by

$$\partial_i^{\mathrm{KZ}} = \frac{\partial}{\partial z_i} + \nu \sum_{j\neq i} (z_i - z_j)^{-1} . \tag{36}$$

We observe that for $\alpha = 1/2$ all K-dependent terms vanish. Since the Knizhnik–Zamolodchikov derivatives satisfy the ordinary Heisenberg algebra $[\partial_i^{\mathrm{KZ}}, z_j] = \delta_{ij}$ one is left, for $\alpha = 1/2$, with the standard realization of the Virasoro algebra.[e] It is worth mentioning that the fact that the K dependence trivializes for $\alpha = 1/2$ in the holomorphic representation for the Calogero Hamiltonian was already observed in Ref. 7 when discussing the interplay between the Calogero model and anyons.

(3) There exists the following important difference between the horizontal sl(2) algebra (27) and the vertical sl(2) algebra (31) with respect to the dependence of the generators on the center of mass and relative coordinates. In the vertical sl(2) the center of mass degrees of freedom decouple from the relative motion degrees of freedom for arbitrary N in view of the relation

$$\sum_{i=1}^{N}\{X_i, Y_i\} = \frac{1}{N}\left(\sum_{i<j}^{N}\{X_i - X_j, Y_i - Y_j\} + \left\{\sum_{i=1}^{N} X_i, \sum_{j=1}^{N} Y_j\right\}\right) . \tag{37}$$

On the contrary, it turns out that for the horizontal sl(2) algebra in the holomorphic representation we have a nontrivial mixture of the center of mass degrees of freedom and the relative motion ones.

The center of mass coordinate y and the relative coordinates $\tilde{z}_i$ are defined by

$$y = \frac{1}{N}\sum_{i=1}^{N} z_i , \quad \tilde{z}_i = z_i - y , \tag{38}$$

$$\frac{\partial}{\partial y} = \sum_{i=1}^{N} \frac{\partial}{\partial z_i} , \quad \frac{\partial}{\partial \tilde{z}_i} = \frac{\partial}{\partial z_i} - \frac{1}{N}\frac{\partial}{\partial y} . \tag{39}$$

Note that $\sum_{i=1}^{N} \tilde{z}_i = \sum_{i=1}^{N} \partial/\partial\tilde{z}_i = 0$ and that $\partial/\partial\tilde{z}_i \tilde{z}_j = \delta_{ij} - 1/N$. The expressions for the generators of the horizontal sl(2) algebra in terms of y and $\tilde{z}_i$, when the arbitrary parameters are fixed according to (29), are given by

[e]One could try to use this observation to give an alternative proof of the existence of the Virasoro algebra with $\alpha = \frac{1}{2}$ for other realizations of the $\mathrm{SH}_N(\nu)$ algebra. Indeed, if there exists an operator U intertwining the holomorphic and some other representation of the $\mathrm{SH}_N(\nu)$ algebra (for example the Calogero representation we used above), then the proof will become trivial for this other representation as well. Unfortunately, we do not know whether there exists such an operator intertwining between the holomorphic and the Calogero representation. Note that, since L_0 is the universal Calogero Hamiltonian, the knowledge of such an operator U would imply in particular an explicit solution of the Calogero model via reduction to an ordinary harmonic oscillator problem.

$$L_1 = \frac{\partial}{\partial y}\,,$$

$$L_0 = y\frac{\partial}{\partial y} + \sum_{i=1}^{N} \tilde{z}_i \frac{\partial}{\partial \tilde{z}_i} + N\left[\lambda_0 + \frac{1}{2} - \alpha + \frac{1}{2}\nu(N-1)\right], \tag{40}$$

$$L_{-1} = y^2\frac{\partial}{\partial y} + 2\sum_{i=1}^{N} y\tilde{z}_i \frac{\partial}{\partial \tilde{z}_i} + \frac{1}{N}\sum_{i=1}^{N} \tilde{z}_i^2 \frac{\partial}{\partial y} + \sum_{i=1}^{N} \tilde{z}_i^2 \frac{\partial}{\partial \tilde{z}_i}$$
$$+ \nu(1-2\alpha)\sum_{i\neq j} \tilde{z}_i K_{ij} + 2Ny\left[\lambda_0 + \frac{1}{2} - \alpha + \frac{1}{2}\nu(N-1)\right].$$

We observe that the relative coordinate operators $\tilde{z}_i, \partial/\partial\tilde{z}_i$ all occur in the z scale-invariant combination $\tilde{z}_i\partial/\partial\tilde{z}_i$ except for three terms in L_{-1}. One of these three terms leads to a nontrivial mixing of the center of mass degrees of freedom and the relative ones in the Calogero model. To get rid of this term, one can consider the limiting procedure where $y \to y$ and $\tilde{z}_i \to \delta\tilde{z}_i$ with $\delta \to 0$, which does not affect the commutation relations of the sl(2) algebra, as can be checked easily by using (40). We observe that after this limiting procedure the relative coordinate-dependent operators always occur in the combination

$$\bar{\lambda} = \sum_{i=1}^{N} \tilde{z}_i \frac{\partial}{\partial \tilde{z}_i} + N\left[\lambda_0 + \frac{1}{2} - \alpha + \frac{1}{2}\nu(N-1)\right]. \tag{41}$$

For this particular degenerate realization the relative coordinate operators thus behave as inner coordinates which only affect the conformal weight.

The above limiting procedure can be applied to the whole Virasoro algebra as well. One may verify that after taking the limit $\delta \to 0$ the generators following from (35) all take the standard form

$$L_{-n} = y^{n+1}\frac{\partial}{\partial y} + \bar{\lambda}(n+1)y^n\,, \tag{42}$$

with the conformal weight $\bar{\lambda}$ given by (41).

Finally, we note that the center of mass coordinate y plays an important role in defining, in a consistent way, the Virasoro generators L_n $(n > 0)$ which involve negative powers of z_i. Specifically, to define inverse powers of z_i one first uses the decomposition $z_i = y + \tilde{z}_i$ and then expands all expressions in powers of the relative coordinates $\tilde{z}_i$, e.g. $z_i^{-1} = y^{-1}\sum_{n=0}^{\infty}(-)^n(\frac{\tilde{z}_i}{y})^n$. In particular, it is convenient to use this approach to derive the limiting form (42). The same approach is used in the analysis of the W_∞ type generalizations which are discussed in Sec. 4.

3. Superextension

It is natural to extend the results obtained in the previous section to the supersymmetric case, thereby extending the Virasoro algebra to a super-Virasoro algebra. Supersymmetric extensions of the Calogero model were recently discussed in Refs. 22 and 7. In the following we will frequently make use of the results of Ref. 7.

Our starting point is the supersymmetric extension of the $SH_N(\nu)$ algebra which is given by the direct product of $SH_N(\nu)$ with the Clifford algebra C_{2N} with generating elements θ_i and $\theta_i^\dagger$:

$$\{\theta_i, \theta_j^\dagger\} = \delta_{ij}\,. \tag{43}$$

The operator K_{ij} is assumed to commute with θ_i and $\theta_i^\dagger$. Note that the fermionic permutation operators K_{ij}^θ can be realized as[f]

$$K_{ij}^\theta = 1 - (\theta_i - \theta_j)(\theta_i^\dagger - \theta_j^\dagger)\,. \tag{44}$$

These operators commute with $a_i, a_i^\dagger$ and have the following standard permutation relations with $\theta_i, \theta_i^\dagger$:

$$\theta_i^{(\dagger)} K_{ij}^\theta = K_{ij}^\theta \theta_j^{(\dagger)}\,, \quad (K_{ij}^\theta)^2 = 1\,. \tag{45}$$

In addition, the operators K_{ij}^θ satisfy the properties (2)–(4). One can also define the total permutation operators

$$K_{ij}^{\text{tot}} = K_{ij} K_{ij}^\theta\,, \tag{46}$$

which exchange both bosonic and fermionic coordinates simultaneously.

In Refs. 22 and 7 the explicitly supersymmetric form of the super-Calogero Hamiltonian H_s was given. In particular, the construction of Ref. 7 was based upon an osp(1,2) supersymmetric extension of the vertical sl(2) algebra:[g]

$$\begin{aligned} H_s &= \{Q, Q^\dagger\}\,, \\ &= \frac{1}{2}\sum_{i=1}^{N}\{a_i, a_i^\dagger\} + \frac{1}{4}\sum_{i=1}^{N}[\theta_i, \theta_i^\dagger] + \frac{1}{2}\nu\sum_{i\neq j} K_{ij}^{\text{tot}}\,, \end{aligned} \tag{47}$$

where

$$Q = \sum_{i=1}^{N}\theta_i^\dagger a_i\,, \quad Q^\dagger = \sum_{i=1}^{N}\theta_i a_i^\dagger\,. \tag{48}$$

The generators $Q, Q^\dagger$ can be interpreted as odd generators of the osp(1,2) superextension of the vertical sl(2) algebra. Note that the last term of (47), when acting

[f]We observe that for $N = 2$ the operator K_{12}^θ equals the Klein operator K which occurs in the discussion of the super-$W_\infty(\lambda)$ algebra.[20] Indeed, we can write $K = 1 - 2\theta\theta^\dagger$, with $\theta^{(\dagger)} \equiv \frac{1}{\sqrt{2}}(\theta_1^{(\dagger)} - \theta_2^{(\dagger)})$. The Klein operator K satisfies $K^2 = 1$ and anticommutes with $\theta^{(\dagger)}$.

[g]We have reintroduced the frequency parameter ω_F of Ref. 7 and taken it to be equal to $\omega_F = -\frac{1}{2}$.

on the subspace of totally symmetric wave functions, reduces to a constant. One may verify that the Hamiltonian

$$H = \frac{1}{2}\sum_{i=1}^{N}\{a_i, a_i^\dagger\} + \frac{1}{4}\sum_{i=1}^{N}[\theta_i, \theta_i^\dagger] \tag{49}$$

satisfies the commutation relations

$$[H, a_i^\dagger] = a_i^\dagger\,, \quad [H, a_i] = -a_i\,, \tag{50}$$

$$[H, \theta_i^\dagger] = -\frac{1}{2}\theta_i^\dagger\,, \quad [H, \theta_i] = \frac{1}{2}\theta_i\,. \tag{51}$$

In order to describe our ansatz for the super-Virasoro generators, it is convenient to introduce the "super-Dunkl derivative" $\mathcal{D}_i$. In Ref. 7 it was shown that there exists a one-parameter class of superderivatives which all fulfill the basic relations

$$\{\mathcal{D}_i, \mathcal{D}_j\} = 2\delta_{ij}D_i^\theta\,, \tag{52}$$

$$\{\mathcal{D}_i, \theta_j\} = \delta_{ij}\,, \tag{53}$$

where D_i^θ is some supersymmetric extension of the ordinary (bosonic) Dunkl derivative. All these derivatives were shown in Ref. 7 to be related to each other by a similarity transform. Remarkably, it turns out that only one particular member of this class leads to the desired superextension of the Virasoro algebra.[h] This particular member is exactly the one which is covariant with respect to the global supersymmetry rules

$$\delta z_i = \theta_i\epsilon\,, \quad \delta\theta_i = \epsilon\,. \tag{54}$$

To be precise, for the realization of the super-Virasoro algebra, we need the special derivative[i]

$$\mathcal{D}_i = \mathcal{D}_i^0 + \nu\sum_{j\neq i}\frac{\theta_{ij}}{z_{ij}}(1 - K_{ij}^{\text{tot}})\,, \tag{55}$$

where

$$\mathcal{D}_i^0 = \frac{\partial}{\partial\theta_i} + \theta_i\frac{\partial}{\partial z_i} \tag{56}$$

is the ordinary supercovariant derivative and where we have introduced the supersymmetric line elements

$$z_{ij} = z_i - z_j - \theta_i\theta_j\,, \quad \theta_{ij} = \theta_i - \theta_j\,. \tag{57}$$

The supercovariant derivatives of these line elements are given by

$$\mathcal{D}_i^0 z_{ij} = \mathcal{D}_j^0 z_{ij} = \theta_{ij}\,, \quad \mathcal{D}_i^0\theta_{ij} = -\mathcal{D}_j^0\theta_{ij} = 1\,. \tag{58}$$

[h]Note that the similarity transform of Ref. 7 does not leave invariant the ansatz we use below.
[i]This derivative equals the super-Dunkl derivative ${}^\alpha\mathcal{D}_i$ of Ref. 7 with $\alpha = \nu$.

A noteworthy property of the super-Dunkl derivative $\mathcal{D}_i$ is that because of the factor of $\theta_i - \theta_j$ in front of $K_{ij}^{\rm tot}$ one can equally well use both K_{ij} and $K_{ij}^{\rm tot}$ in its definition. The expression for the bosonic derivative D_i^θ follows from the anticommutator of two super-Dunkl derivatives and is given by

$$D_i^\theta = \frac{\partial}{\partial z_i} + \nu \sum_{l \neq i} \left[\frac{1}{z_{il}}(1 - K_{il}^{\rm tot}) - \frac{\theta_{il}}{z_{il}} K_{il}^{\rm tot} \left(\mathcal{D}_i - \mathcal{D}_l - \nu \sum_{j \neq i, j \neq l} \frac{\theta_{lj}}{z_{lj}} K_{lj}^{\rm tot} \right) \right] . \tag{59}$$

One easily finds that

$$[\mathcal{D}_i, z_j] = \delta_{ij} \left(\theta_i + \nu \sum_{k \neq i = 1}^{N} \theta_{ik} K_{ik} \right) - \nu \theta_{ij} K_{ij} . \tag{60}$$

This formula is a special case of a more general formula which gives the commutator of $\mathcal{D}_i$ with an arbitrary function $f(z, \theta)$ of a single superargument (z, θ). By the use of the convention $f_i \equiv f(z_i, \theta_i)$ and the abbreviation

$$y_{ij} \equiv \frac{\theta_{ij}}{z_{ij}} , \tag{61}$$

the general formula is given by

$$[\mathcal{D}_i, f_i] = \mathcal{D}^0(f)_i + \nu \sum_{i \neq j} (f_i - f_j)\, y_{ij} K_{ij}^{\rm tot} , \tag{62}$$

$$[\mathcal{D}_i, f_j] = -\nu (f_i - f_j)\, y_{ij} K_{ij}^{\rm tot} , \quad i \neq j , \tag{63}$$

$$\sum_i [\mathcal{D}_i, f_i] = \sum_i \mathcal{D}^0(f)_i . \tag{64}$$

For the convenience of the reader we give below some useful identities that are obeyed by the quantities y_{ij}:

$$\begin{gathered} y_{ij} = y_{ji} , \quad y_{ij} y_{kl} = -y_{kl} y_{ij} , \\ y_{il} y_{lj} + y_{jl} y_{ij} + y_{ij} y_{il} = 0 , \\ \mathcal{D}_i^0 y_{ij} = -\mathcal{D}_j^0 y_{ij} = \frac{1}{z_{ij}} . \end{gathered} \tag{65}$$

We now proceed with our ansatz for the (super)generators of the $(N = 2)$ super-Virasoro algebra which is given by

$$L_\xi = \sum_i \left[\mathcal{D}_i \xi_i \mathcal{D}_i - \frac{1}{2} \mathcal{D}^0(\xi)_i \mathcal{D}_i + \frac{1}{2} \lambda \xi_i' \right] , \tag{66}$$

$$Q_\epsilon = \sum_i \epsilon_i \mathcal{D}_i + \lambda \sum_i \mathcal{D}^0(\epsilon)_i , \tag{67}$$

where $\xi(z,\theta)$ and $\epsilon(z,\theta)$ are arbitrary commuting parameters and $f'(z,\theta) \equiv \frac{\partial}{\partial z} f(z,\theta)$.

One can check that the following (anti)commutation relations are true:

$$\{Q_{\epsilon_1}, Q_{\epsilon_2}\} = L_{\xi_{1,2}}\,, \tag{68}$$

$$[L_\xi, Q_\epsilon] = Q_{\bar{\epsilon}}\,, \tag{69}$$

where

$$\xi_{1,2} = 2\epsilon_1\epsilon_2\,, \tag{70}$$

$$\bar{\epsilon} = \xi\epsilon' - \frac{1}{2}\epsilon\xi' + \frac{1}{2}\mathcal{D}^0(\xi)\mathcal{D}^0(\epsilon)\,. \tag{71}$$

The proof of the anticommutation relation (68) is relatively simple. To illustrate how it works we consider the case with $\lambda = 0$. A straightforward calculation gives

$$Q_\epsilon Q_\epsilon = \sum_{i,j} \epsilon_i \mathcal{D}_i \epsilon_j \mathcal{D}_j = \sum_{i,j} (\epsilon_i\epsilon_j \mathcal{D}_i\mathcal{D}_j + \epsilon_i[\mathcal{D}_i, \epsilon_j]\mathcal{D}_j)\,. \tag{72}$$

Taking into account the fact that the parameter ϵ is commuting, the basic anticommutation relation (52) of the super-Dunkl derivatives and the formulae (62) and (63), one obtains after some simple algebra

$$Q_\epsilon Q_\epsilon = \sum_i [\epsilon_i^2 \mathcal{D}_i^2 + \epsilon_i \mathcal{D}_i^0(\epsilon_i)\mathcal{D}_i] + \nu \sum_{i\neq j} \epsilon_i^2 y_{ij} K_{ij}^{\mathrm{tot}} (\mathcal{D}_i - \mathcal{D}_j)\,. \tag{73}$$

Analogously one finds that

$$\sum_i \mathcal{D}_i \epsilon_i^2 \mathcal{D}_i = \sum_i [\epsilon_i^2 \mathcal{D}_i^2 + \mathcal{D}_i^0(\epsilon_i^2)\mathcal{D}_i] + \nu \sum_{i\neq j} \epsilon_i^2 y_{ij} K_{ij}^{\mathrm{tot}} (\mathcal{D}_i - \mathcal{D}_j)\,. \tag{74}$$

Combining (73) with (74) proves the relation (68) for $\lambda = 0$. It is simple to generalize this relation to nonvanishing values of λ.

The proof of the $[L_\xi, Q_\epsilon]$ commutator is much more involved and is described in App. B. Let us note that as in the bosonic case the fact that the generators Q and L form a closed super-Virasoro algebra is representation-independent, i.e. it follows only from the basic commutation relations (52), (53) and (60). Note that from the relations (68) and (69) it follows that the L generators satisfy the commutation relations of the Virasoro algebra.

Like in the bosonic case L_0 can be identified with a Z grading operator by adjusting the λ parameter as follows:

$$\lambda = \lambda_0 + \nu N^{-1} \sum_{i\neq j} (K_{ij}^{\mathrm{tot}} - 1)\,. \tag{75}$$

The generators L_1, L_0 and L_{-1} of the sl_2 subalgebra of Virasoro which correspond respectively to the parameters $\xi = 1$, z, and z^2 in (66) then take the forms

$$L_1 = \sum_{i=1}^{N} \frac{\partial}{\partial z_i}\,, \tag{76}$$

$$L_0 = \sum_i z_i \frac{\partial}{\partial z_i} + \frac{1}{2}\sum_i \theta_i \frac{\partial}{\partial \theta_i} + \frac{1}{2}\lambda_0 N\,, \tag{77}$$

$$\begin{aligned} L_{-1} = {} & \sum_i z_i^2 \frac{\partial}{\partial z_i} + \sum_i z_i \theta_i \frac{\partial}{\partial \theta_i} + \lambda_0 \sum_{i=1}^{N} z_i - \nu \sum_{i\neq j} z_i K_{ij}^{\mathrm{tot}} \\ & + \nu N^{-1} \sum_{i\neq j} K_{ij}^{\mathrm{tot}} \sum_{l=1}^{N} z_l + \nu^2 \sum_{j\neq k\neq i} z_i^2 y_{ij} K_{ij}^{\mathrm{tot}} y_{jk} K_{jk}^{\mathrm{tot}}\,. \end{aligned} \tag{78}$$

The ν^2 contribution to L_0 from the last term in the representation (91) for the Virasoro generators vanishes due to the identity

$$z_l y_{il} y_{lj} + z_j y_{jl} y_{ij} + z_i y_{ij} y_{il} = 0\,. \tag{79}$$

The fermionic coordinates in L_0 appear only as a shift of the conformal parameter λ_0 which involves the fermionic number operator $\sum_i \theta_i \frac{\partial}{\partial \theta_i}$. This is not the case for the L_1 generator, which involves nontrivial mixings between the relative and fermionic coordinates.

An interesting distinction between the bosonic and the fermionic case is that the supergenerators involve only one independent free parameter, the conformal weight λ. Due to the identity (64) a term of the form $\sum_i \mathcal{D}_i \epsilon_i$ can be rewritten in terms of $\sum_i \epsilon_i \mathcal{D}_i$ modulo a shift in the parameter λ. Therefore, in the supersymmetric case, there is no room for a second free parameter, like the parameter α in the bosonic case.

When restricted to the subspace of θ-independent functions, the above generators of the sl_2 subalgebra of the super-Virasoro algebra coincide with the bosonic generators (27) for $\alpha = 1$ [and with the conformal weight parameters identified as $\lambda(\text{bosonic}) - 1/2 = 1/2\lambda(\text{fermionic})$] except for the ν^2-dependent term in L_{-1}. This term contains an explicit θ dependence. Thus, for $\nu \neq 0$, one cannot truncate the sl_2 subalgebra of the super-Virasoro algebra to the sector of operators acting on θ-independent functions.

The super-Virasoro algebra under consideration is formulated here in terms of $N = 1$ superfield parameters analogous to the one-particle formulation given in Ref. 20. The algebra, however, is $(N = 2)$-supersymmetric, as can easily be seen from the component analysis: the superfield generators L_ξ and Q_ϵ involve one spin 2 current, *two* spin 3/2 currents (one from L_ξ, and another one from Q_ϵ), and one spin 1 current (the θ component in Q_ϵ). In particular, the u(1) component of the spin 1 current has the form

$$J_0 = \sum_i \theta_i \frac{\partial}{\partial \theta_i} - \nu \sum_{i \neq j} \frac{\theta_i \theta_j}{z_{ij}} (1 - K_{ij}^{\text{tot}}) . \tag{80}$$

Along with the corresponding supergenerators the above spin 2 and spin 1 generators form the "horizontal" osp(2|2) subalgebra of the full super-Virasoro algebra. This algebra should be distinguished from the "vertical" osp(2|2) algebra which was considered in Ref. 7.

4. Conclusions

In this paper we have shown that based on the Calogero-model-inspired algebras $\mathrm{SH}_N(\nu)$ and their superextensions one can construct new realizations of the (super-)Virasoro algebra analogous to the standard realization by (super)vector fields based on the ordinary Heisenberg algebras (or, equivalently, the algebra of differential operators). The algebra $\mathrm{SH}_N(\nu)$ can be regarded as the associative algebra with generating elements $a_i^\dagger$, a_j and K_{ij}. One can consider the same algebra but now with the (super)commutator as the product law. This leads to infinite-dimensional Lie (super)algebras, which we denote by $W_{N,\infty}(\nu)$ in analogy with the construction of ordinary W_∞ type algebras via commutators of elements of the enveloping algebra of the Heisenberg algebra.[12,23,24] Due to the results of this paper these algebras contain the Virasoro subalgebra and therefore can be regarded as an extended conformal algebra, thereby justifying the name "W_∞ type algebra." Note that the $W_{1+\infty}$ algebra can be identified with the algebra $W_{1,\infty}(\nu)$ (the parameter ν drops out for $N = 1$).

The $W_{N,\infty}(\nu)$ ($N \geq 2$) algebras are rather large algebras involving an infinite number of higher spin generators of each spin. An interesting question is whether there exist truncations of the $W_{N,\infty}(\nu)$ algebra that still contain the Virasoro algebra. There indeed exists such a truncated algebra. This subalgebra is spanned by the symmetric elements w which commute with the permutation operators, i.e.

$$[K_{ij}\,, w] = 0 . \tag{81}$$

We denote this subalgebra of $W_{N,\infty}(\nu)$ by $\tilde{W}_{N,\infty}(\nu)$. Since the Virasoro generators themselves obey the symmetry conditions (23), $\tilde{W}_{N,\infty}(\nu)$ contains the Virasoro algebra as a subalgebra. The algebra $\tilde{W}_{N,\infty}(\nu)$ contains infinitely many higher spin currents of each spin.

A natural question to ask is whether $W_{1+\infty}$ is a proper subalgebra of $W_{N,\infty}(\nu)$. The answer seems to be negative. So far our attempts to embed $W_{1+\infty}$ into $W_{N,\infty}(\nu)$ as a proper subalgebra have not been successful.

The above results can be naturally extended to the supersymmetric case. One can define the superalgebras $SW_{N,\infty}(\nu)$ and $S\tilde{W}_{N,\infty}(\nu)$ based on the superextensions considered in Sec. 3. The superalgebra $S\tilde{W}_{N,\infty}(\nu)$ is spanned by the symmetric elements obeying the conditions

$$[K_{ij}^{\text{tot}}, w] = 0\,. \tag{82}$$

We believe that the generalized W_∞ type algebras introduced above may have interesting applications in the context of conformal field theory, integrable systems and the quantum Hall effect. We hope to discuss some of these applications in a future publication.

Acknowledgments

We thank L. Brink, S. Isakov, J. Leinaas and A. Polychronakos for useful discussions. The work of E. B. has been made possible by a fellowship of the Royal Netherlands Academy of Arts and Sciences (KNAW). The research of M. V. described in this publication was made possible in part by Grant #MQM000 from the International Science Foundation. M. V. would like to thank Groningen University for hospitality.

Appendix A. Proof of the Identities (14)–(16)

To prove the identities (14)–(16) we need a particular case of (17) when $f(a^\dagger)$ is a function of one argument $a_i^\dagger$ for some fixed i, i.e. $f(a^\dagger) = \phi(a_i^\dagger)$. For this particular case we easily derive from (17)

$$\begin{aligned}[a_i, \phi(a_j^\dagger)] = \delta_{ij}\phi'(a_i^\dagger) + \nu\delta_{ij}\sum_{l=1}^{N}(a_i^\dagger - a_l^\dagger)^{-1}[\phi(a_i^\dagger) - \phi(a_l^\dagger)]K_{il} \\ - \nu(a_i^\dagger - a_j^\dagger)^{-1}[\phi(a_i^\dagger) - \phi(a_j^\dagger)]K_{ij}\,,\end{aligned} \tag{83}$$

where $\phi'(x) = \frac{\partial}{\partial x}\phi(x)$. Note that the commutation relations (1) are themselves a particular case of (83) for $\phi(x) = x$.

The proof of (14) and (15) is straightforward by observing that the two ν-dependent terms coming from the right hand side of (83) cancel against each other after the substitution of (83) into (14) and (15). The ν-independent part of the resulting expression gives the right hand side of (14) and (15).

To prove the identity (16) it is convenient to decompose the left hand side of (16),

$$X \equiv \sum_{i,j=1}^{N}\left[[a_i, \xi_1(a_i^\dagger)], [a_j, \xi_2(a_j^\dagger)]\right], \tag{84}$$

into three parts, $X^{(0)}, X^{(1)}$ and $X^{(2)}$, which are, respectively, independent of ν, linear in ν and quadratic in ν. Due to (83) we obviously have $X^{(0)} = 0$. Using the shorthand notation $\xi_{1i} = \xi_1(a_i^\dagger)$ and $\xi_{2i} = \xi_2(a_i^\dagger)$ we can write

$$X^{(1)} = \nu \sum_{i=1}^{N} [\xi'_{1i}, \sum_{l\neq i} (a_i^\dagger - a_l^\dagger)^{-1} (\xi_{2i} - \xi_{2l}) K_{il}] - (1 \leftrightarrow 2)$$

$$= \nu \sum_{l\neq i} (\xi'_{1i} - \xi'_{1l})(a_i^\dagger - a_l^\dagger)^{-1} (\xi_{2i} - \xi_{2l}) K_{il} - (1 \leftrightarrow 2)\,, \qquad (85)$$

which also vanishes because the summand is antisymmetric under the interchange $i \leftrightarrow l$.

Finally, for $X^{(2)}$ one gets from (83)

$$X^{(2)} = \nu^2 \sum_{i,l,n} [(a_i^\dagger - a_l^\dagger)^{-1} (\xi_{1i} - \xi_{1l}) K_{il}, (a_i^\dagger - a_n^\dagger)^{-1} (\xi_{2i} - \xi_{2n}) K_{in}]\,. \qquad (86)$$

The contribution to the summation of the terms with $l = n$ vanishes, so that one is left with

$$X^{(2)} = \nu^2 \sum_{i\neq l, i\neq n, l\neq n} \Big[(a_i^\dagger - a_l^\dagger)^{-1} (\xi_{1i} - \xi_{1l}) (a_l^\dagger - a_n^\dagger)^{-1} (\xi_{2l} - \xi_{2n}) K_{il} K_{in}$$

$$- (a_i^\dagger - a_n^\dagger)^{-1} (\xi_{2i} - \xi_{2n}) (a_n^\dagger - a_l^\dagger)^{-1} (\xi_{1n} - \xi_{1l}) K_{in} K_{il} \Big]\,. \qquad (87)$$

After replacing $l \leftrightarrow n$ in the second term one can write $X^{(2)}$ as

$$X^{(2)} = \nu^2 \sum_{i\neq l, i\neq n, l\neq n} (a_i^\dagger - a_l^\dagger)^{-1} (a_l^\dagger - a_n^\dagger)^{-1} (a_i^\dagger - a_n^\dagger)^{-1}$$

$$\times [(a_i^\dagger - a_n^\dagger)(\xi_{1i}\xi_{2l} + \xi_{1l}\xi_{2n} - \xi_{1i}\xi_{2n}) - 1 \leftrightarrow 2] K_{il} K_{in}\,. \qquad (88)$$

Finally, using the cyclic property (2) of the elementary permutation generators one easily verifies, by permuting the summation indices i, j, l, that the $(a_i^\dagger - a_n^\dagger)$ coefficient in the second line can be replaced by $1/3[(a_i^\dagger - a_n^\dagger) + (a_l^\dagger - a_i^\dagger) + (a_n^\dagger - a_l^\dagger)]$, which is identically zero. This concludes the proof of the identity (16).

In addition to the identities proven above there also exist the following useful identities which can be proven analogously:

$$\sum_{i=1}^{N} \Big\{ [a_i, (\xi_1(a_j^\dagger)], \xi_2(a_i^\dagger) \Big\} = 2\xi_2(a_j^\dagger)\xi'_1(a_j^\dagger)\,, \qquad (89)$$

$$\sum_{i,j=1}^{N} \Big[[a_i, \xi_1(a_j^\dagger)], [a_j, \xi_2(a_i^\dagger)] \Big] = 0\,. \qquad (90)$$

Appendix B. The Sketch of Calculation of $[L_\xi, Q_\epsilon]$

We outline below the main steps of our calculation of the $[L_\xi, Q_\epsilon]$ commutator for the case with $\lambda = 0$.

It is convenient to work with the following form for the ansatz of the L_ξ generator, which can easily be obtained from (66) with the aid of the formula (59):

$$L_\xi = \sum_i \xi_i D_i^{\theta,0} + \frac{1}{2}(\mathcal{D}_i^0 \xi_i)\mathcal{D}_i + \nu^2 \sum_{j\neq k\neq i} \xi_i y_{ij} K_{ij}^{\rm tot} y_{jk} K_{jk}^{\rm tot}, \tag{91}$$

with

$$D_i^{\theta,0} \equiv \frac{\partial}{\partial z_i} + \nu \sum_{j\neq i} \frac{1}{z_{ij}}(1 - K_{ij}^{\rm tot}). \tag{92}$$

As the first stage we find for the commutator

$$\begin{aligned}
[L_\xi, Q_\epsilon] = \sum_{i,j} \Big(& \xi_i [D_i^{\theta,0}, \epsilon_j]\mathcal{D}_j + \xi_i \epsilon_j [D_i^{\theta,0}, \mathcal{D}_j] - \epsilon_i [\mathcal{D}_i, \xi_j] D_j^{\theta,0} \\
& + \frac{1}{2}(\mathcal{D}_i^0 \xi_i)[\mathcal{D}_i, \epsilon_j]\mathcal{D}_j + \frac{1}{2}\epsilon_j (\mathcal{D}_i^0 \xi_i)\{\mathcal{D}_i, \mathcal{D}_j\} - \frac{1}{2}\epsilon_j \{\mathcal{D}_j, (\mathcal{D}_i^0 \xi_i)\}\mathcal{D}_i \Big) \\
& + \nu^2 \sum_{i\neq j\neq k} \sum_l [\xi_i y_{ij} K_{ij}^{\rm tot} y_{jk} K_{jk}^{\rm tot}, \epsilon_l \mathcal{D}_l]\,.
\end{aligned} \tag{93}$$

We first consider the terms in the lowest order in ν which are of the form $\xi\epsilon'\mathcal{D}_i, \epsilon\xi'\mathcal{D}_i, (\mathcal{D}_i^0\xi_i)(\mathcal{D}_i^0\epsilon_i)\mathcal{D}_i$ and $\epsilon_i(\mathcal{D}_i^0\xi_i)D_i^{\theta,0}$. The first three types of terms gives us the required supersymmetry transformation with the parameter (70). The last type of term cancels.[j] We next consider the remaining terms linear in ν. They come only from the first and second lines in (93). We observe that all ν-dependent terms in the second line of (93) are proportional to $(\mathcal{D}_i^0\xi_i)$, whereas all terms in the first line are proportional to ξ_i. This implies that the remaining ν-dependent terms occurring in the first and second lines should cancel independently. The second line in (93) gives rise to terms of the form[k]

$$\nu \frac{\theta_{ij}}{z_{ij}} (\mathcal{D}_i^0 \xi_i)\epsilon_i K_{ij}^{\rm tot} \mathcal{D}_i\,. \tag{94}$$

These terms can be shown to cancel amongst each other without the need to write out explicitly the extra ν^2-dependent terms occurring inside $\mathcal{D}_i$. Therefore, this cancellation also involves a class of terms quadratic in ν.

[j] In doing the calculation there is no need to write out the derivatives $\mathcal{D}_i$ and $D_i^{\theta,0}$ in terms of a ν-independent part and terms linear in ν. Therefore, strictly speaking, our calculation also takes care of a class of ν^2-dependent terms in the commutator.

[k] We give only the general structure of the terms. We use a notation where the specific name of the indices is irrelevant.

We next consider the terms linear in ν coming from the first line in (93). The following three types of terms linear in ν occur:

$$\nu\frac{\theta_{ij}}{(z_{ij})^2}\xi_i\epsilon_j(1-K_{ij}^{\rm tot}),\quad \nu\frac{\theta_{ij}}{z_{ij}}\xi_i\epsilon_j K_{ij}^{\rm tot}\frac{\partial}{\partial z_i},\quad \nu\frac{1}{z_{ij}}\xi_i\epsilon_j K_{ij}^{\rm tot}\mathcal{D}_i^0\,. \tag{95}$$

All three types of terms can be shown to cancel amongst each other.

We now consider the terms quadratic in ν. In the second line of (93), only the anticommutator in the second term leads to a ν^2-dependent term which has not yet been considered. This term cancels against a similar $(\mathcal{D}_i^0\xi_i)$-dependent term coming from the commutator in the third line of (93). This finishes the discussion of the second line in (93): all terms either have been canceled or contribute to the supersymmetry parameter $\bar{\epsilon}$ given in (70). All remaining ν^2-dependent terms come from the first and second lines in (93). The following two types of terms occur:

$$\nu^2\frac{\theta_{ij}}{(z_{ij})^2}\xi_i\epsilon_j K_{ij}^{\rm tot}\,,\quad \nu^2\frac{\theta_{ij}}{(z_{ij})^2}\xi_i\epsilon_j K_{ij}^{\rm tot}K_{jk}^{\rm tot}\,. \tag{96}$$

The terms linear in $K_{ij}^{\rm tot}$ come from only the first line in (93) and it can be shown that they cancel amongst each other. We next consider the terms quadratic in $K_{ij}^{\rm tot}$. Once we have moved the permutation operators to the right we may distinguish between two types of terms which should cancel independently of each other. Terms of the first type are proportional to $\xi_i\epsilon_i$, i.e. ξ and ϵ have the same index; terms of the second type are proportional to $\xi_i\epsilon_j$ with $i\neq j$. Both types of terms can be further subdivided into either terms with a summation over three different indices $i\neq j\neq k$ or terms with a summation over two different indices $i\neq j$. A straightforward but somewhat tedious calculation shows that all types of terms cancel amongst each other.

Finally, we consider the terms trilinear in ν. They come from only the commutator in the third line of (93) and are given by

$$\nu^3\sum_{i\neq j\neq k}\sum_{l\neq m}\Big[\epsilon_l(y_{lm}K_{lm}^{\rm tot}),\xi_i(y_{ij}K_{ij}^{\rm tot})(y_{jk}K_{jk}^{\rm tot})\Big]\,. \tag{97}$$

We first consider the case where either l or m is equal to i,j or k but not both. The terms in the commutator corresponding to this case can always be written as the following three types of terms:[1]

$$\nu^3\sum_{i\neq j\neq k\neq l}\Big(\epsilon_i\xi_i\ y_{ij}y_{ik}y_{il}\times K^3\,,\quad \epsilon_i\xi_j\ y_{ij}y_{jk}y_{jl}\times K^3\,,\quad \epsilon_i\xi_j\ y_{ij}y_{jk}y_{il}\times K^3\Big)\,. \tag{98}$$

To do this one must make use of the y identities given in (65). A tedious calculation shows that all three types of terms cancel. Finally, we consider the case where both

[1]The specific name of the indices in this formula is important now.

l and m are equal to i, j or k. It can be easily shown that this case always leads to terms which identically vanish.

It is straightforward to extend the calculation of the $[L_\xi, Q_\epsilon]$ commutator to nonzero values of λ.

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